IMPACT OF FLUID LOADING ON THE RADIATED SOUND POWER OF VIBRATING PLATES

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1. OVERVIEW

In active structures applications the fluid-structure interaction is often an important problem, especially for the problem of vibration and noise reduction. The impact of fluid loading on the vibration of a structure is essential in submarine applications. For fluid-gas coupling, the fluid loading is often neglected, but this neglection can result in poor performance of vibration and noise control.

This paper is based on the Master's Thesis of the author which was written at the MIT Active Materials and Structures Laboratory in conjunction with the Institut für Aerodynamik und Gasdynamik (IAG) at the University of Stuttgart.

2. INTRODUCTION

The reduction of radiated noise of structure-born sound is an important goal in the smart materials and structures research. Usually the sound radiation is modelled neglecting the fluid loading, which is in part possible for structures in many gases. For structures submerged in liquids, such as underwater buildings, the radiated noise and the reaction force of the surrounding liquid on the structure can no longer be neglected.

Active Noise Control (ANC) is based on sound measurements and anti-noise speakers deployed in the surroundings. ANC only provides local control of noise. Alternatively, Active Structural Acoustic Control (ASAC) minimizes the radiated noise by means of structural control. ASAC provides global control of the radiated noise and doesn't require any sensors or actuators (microphones and speaker) deployed in the surroundings of the structure. For ASAC an accurate coupled fluid-structure interaction model using only data local to the structure is needed.

3. PLATES

The flat plate has been chosen as an example of a vibrating structure. The experiment at the MIT Active Materials and Structures Laboratory has been set up in an anechoic chamber as shown in figure 1.

Figure 1: The experimental setup

The coordinate systems used in this work are shown in figure 2. The origin is the center of the plate. Two cartesian coordinate systems (x, y, z) and (x_0, y_0, z) are defined together with a spherical coordinate system (ϕ, θ, R). (x, y), (x_0, y_0) and ϕ are coordinates in plane, z is the cartesian coordinate out of plane, θ is the angle out of plane, and R the radius of the spherical coordinates.

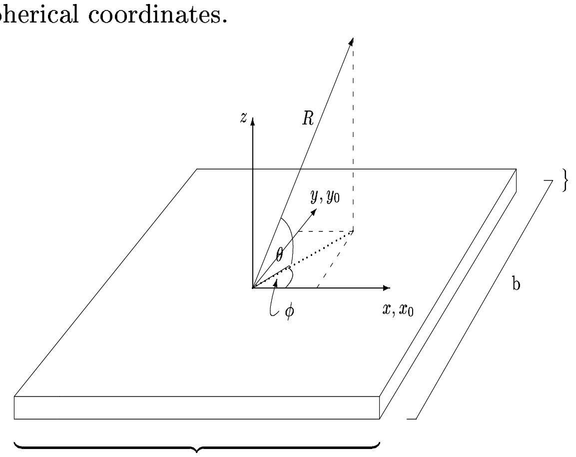

Figure 2: Coordinate systems

In this work scalars will be denoted by plain characters while vectors will be denoted using lowercase bold characters and matrices will be denoted using uppercase bold characters.

4. THE PLATE MODEL

Assuming a flat plate, isotropic and homogeneous material, small deflections, constant volume and surface area and the plate lying in the $x - y(x_0, y_0)$ plane, the governing equation for the plate deflections is given by Rayleigh [7] as

$$D\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)\eta(x,y,t) + \rho h \frac{\partial^2 \eta(x,y,t)}{\partial t^2} = p_e(x,y,t) \qquad (1)$$

where $D = Eh^3 / \left[12(1-\nu^2)\right]$, E is Young's modulus, h the plate thickness, ν Poisson's ratio, ρ the plate density, and $\eta(x,y,t)$ the plate normal deflection.

The boundary conditions for the on all sides simply supported plate, as used in the experimental setup, are given by

$$\begin{aligned} \eta\big|_{boundary} &= 0 \\ \eta''\big|_{boundary} &= 0 \end{aligned} \qquad (2)$$

The eigenmodes of an on all four sides simply supported plate are shown in figure 3.

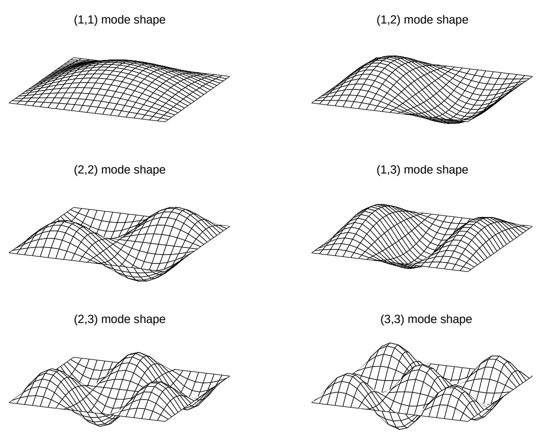

Figure 3: Plate eigenmodes

The mode shapes η_{ij} and eigenfrequencies f_{ij} have been calculated using analytical formulations given by Blevins [2]

$$f_{ij} = \frac{\lambda_{ij}^2}{2\pi a^2}\left[\frac{Eh^3}{12\gamma(1-\nu^2)}\right]^{1/2} \qquad (3)$$

where the coefficients λ_{ij} and the mass per unit area γ are given by

$$\begin{aligned} \lambda_{ij} &= \pi^2\left[i^2 + j^2\left(\frac{a}{b}\right)^2\right] && (4) \\ \gamma &= \rho \cdot h && (5) \end{aligned}$$

5. MECHANICAL IMPEDANCE

Impedances are complex ratios describing energy variations in dynamic systems. The often used electrical impedance is defined as the ratio of voltage over current, whereas the mechanical impedance is defined as the ratio of force over velocity. In table 1 the analogies between electrical and mechanical impedances are shown.

mechanical	**electrical**	**impedance**
mass	inductor	$\mathcal{Z} = -j\mathcal{X}$
spring	capacitor	$\mathcal{Z} = +j\mathcal{X}$
dashpot	resistor	$\mathcal{Z} = \mathcal{R}$

Table 1: Analogies between electrical and mechanical impedances

The real valued part, the resistance $\mathcal{R}$, of the impedance $\mathcal{Z} = \mathcal{R} + j\mathcal{X}$ stands for energy dissipation whereas the imaginary part, the reactance $\mathcal{X}$, denotes energy storage.

In general, impedances are complex valued frequency response functions, they depend on the frequency and are only defined for harmonic excitations. Impedances are ratios of two phasors whose product is power (such as e.g. force and velocity).

Given an equation of motion

$$F = m\ddot{x} + c\dot{x} + kx \qquad (6)$$

and assuming harmonic excitation with $\dot{x} = j\omega x$ and $\ddot{x} = -\omega^2 x$ the mechanical impedance is

$$\mathcal{R} + j\mathcal{X} = \frac{F}{\dot{x}} = c + j \cdot \left(\frac{1}{\omega}k - \omega m\right) \qquad (7)$$

6. FLUID EQUATIONS

For acoustic wave propagation phenomena the following assumptions have proven to be reasonable:

- no mean flow
- inviscid fluid

- adiabatic wave propagation process
- compressible fluid
- small deviations of pressure and density from the ambient values

For locations very far from the source, the wave propagation can be considered to be a plane wave, where pressure and velocity are in phase, propagating in an infinite acoustic medium. In this case, the specific acoustic impedance only contains a real-valued resistance term whose magnitude is equal to the product of the equilibrium mass density of the fluid and the speed of sound ($\rho_\infty c$). This product is called the *characteristic acoustic impedance* of the fluid

$$\frac{p}{u}\Big|_{r\to\infty} = \rho_\infty c \tag{8}$$

Typical fluid properties are shown in table 2.

	air	**water**	dimension
mass density ρ_∞	1.15	1000	$[kg/m^3]$
speed of sound c	350	1500	$[m/s]$
char. impedance $\mathcal{Z}$	400	$1.5 \cdot 10^6$	$[kg/m^2 s]$

Table 2: Fluid properties at T=30°C,p_∞=1013 mbar

7. DERIVATION OF ACOUSTIC INTEGRALS

Starting with the Euler equation

$$dm\,\frac{d\mathbf{u}}{dt} = -\nabla p\, dV \tag{9}$$

the equation of continuity

$$\operatorname{div}(\rho\mathbf{u}) + \frac{\partial \rho}{\partial t} = 0 \tag{10}$$

and the equation for the speed of sound

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\mathrm{S}=const.} \tag{11}$$

the formulation for the divergence of the fluid velocity for small disturbances in the density results in

$$\nabla \cdot \mathbf{u} = -\frac{1}{\rho c^2}\frac{\partial p}{\partial t} \tag{12}$$

7.1 The Helmholtz equation

Assuming a separable and time-harmonic solution

$$p(\mathbf{r},t) = \psi(\mathbf{r}) \cdot e^{j\omega t} \tag{13}$$

of the linear wave equation

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{14}$$

the second time derivative of the pressure can be rewritten as

$$\frac{\partial^2 p}{\partial t^2} = -\omega^2 p(\mathbf{r},t) \tag{15}$$

Using the wave number $k = \omega/c$, the linear wave equation can then be transformed into the homogeneous Helmholtz equation

$$(\nabla^2 + k^2)p = 0 \tag{16}$$

7.2 The Green's function

The solution $g(\mathbf{r} - \mathbf{r}_0)$ of the inhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)g(\mathbf{r} - \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0) \tag{17}$$

where δ is the three-dimensional Dirac (delta) function, is called a Green's function. The construction of the solution g is possible, if the surface can be completely described by specifying one coordinate of a set of three rectangular coordinates, and the solution to the wave equation is separable in this coordinate system. This requires for a surface of infinite extent, such as an infinite plate or an infinite cylinder. One can easily assume a plate of infinite extent with $\eta = 0\ \forall\ (x_0, y_0) \notin S_0$, which is a plate of finite extent S_0 in a rigid boundary.

The Green's function solving the inhomogeneous Helmholtz equation is given by

$$g_\omega(\mathbf{r}) = \frac{e^{jkR}}{4\pi R} + \chi \tag{18}$$

The term χ can be determined, such that the normal gradient of this Green's function on the plate vanishes. It can be shown [6] that this term is of the same form as the first term, but accounting for the reflected wave on the surface, which is modelled by a fictive mirror source inside the plate. This yields

$$g_\omega(\mathbf{r}) = \frac{e^{jkR}}{4\pi R} + \frac{e^{jkR'}}{4\pi R'} \tag{19}$$

where

$$\begin{aligned} R &= \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} \\ R' &= \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z+z_0)^2} \end{aligned}$$

For an acoustic source on the boundary surface, the source and the mirror source get identical and the two terms of the Green's function sum up to

$$g_\omega(R) = \frac{e^{jkR}}{2\pi R} \tag{20}$$

7.3 The Helmholtz integral equation

Solving the homogeneous Helmholtz equation (16) and the inhomogeneous Helmholtz equation (17) for the Laplacian of the fluid pressure and the Green's function respectively, one obtains

$$\begin{aligned} \nabla^2 p &= -k^2 p \\ \nabla^2 G &= \delta - k^2 G \end{aligned} \tag{21}$$

Now constructing a function

$$F = p\nabla G - G\nabla p \tag{22}$$

and applying Gauß's integral theorem

$$\int_V \nabla \cdot F \, dV = -\int_S f \cdot \mathbf{n} \, dS \tag{23}$$

to the function F, keeping in mind that by definition the Dirac delta function is given by

$$\int_V p \cdot \delta \, dV = p \cdot \epsilon \tag{24}$$

where the paramater $\epsilon = 1$ for a point in dV, but not on the plate surface, $\epsilon = 2$ for a point on the plate surface, otherwise ϵ being equal to zero, the Helmholtz integral is obtained.

$$p \cdot \epsilon = -\int_S \left(p \frac{\partial G}{\partial n} - G \frac{\partial p}{\partial n} \right) dS \tag{25}$$

This equation can be split into two surface integrals over different areas, the active area of the plate and an surrounding spherical surface of infinite radius. It can be shown [5] that the integral over the spherical surface vanishes and only the integral over the active plate surface has to be evaluated.

Evaluating the Euler equation (9) at the active plate surface, using $dm = \rho \, dV$, yields

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p \tag{26}$$

The normal derivative of the acoustic pressure is then given in terms of the plate surface normal acceleration $\ddot{\eta} = \dot{u}_z$ as

$$\frac{\partial p}{\partial n} = -\rho \ddot{\eta} \tag{27}$$

Substituting this boundary condition into the Helmholtz integral equation and using the free-space Green's function g, the Helmholtz integral equation can be written as

$$p = -2 \int_S \left(p \frac{\partial g}{\partial n} - \rho \ddot{\eta} g \right) dS \tag{28}$$

7.4 The acoustic impedance

The specific acoustic impedance at a point i due to a source in j is defined as

$$\mathcal{Z}_{ij} = \frac{p_{i|j}}{u_j} \tag{29}$$

where $p_{i|j}$ is the fluid pressure at point i due to the fluid velocity u_j at point j, here, in the considered case, the surface velocity of a point j on the vibrating plate surface.

The force on a surface can be directly computed by evaluating the acoustic impedance

$$F_{i|j} = p_{i|j} dS = \mathcal{Z}_{ij} \cdot u_j \, dS \tag{30}$$

Writing the acoustical impedance in terms of a real-valued resistance $\mathcal{R}$ and a pure imaginary reactance $\mathcal{X}$, a formulation which can be used as an external forcing term in mechanical plate models is obtained:

$$F_{i|j} = \rho c \left(\mathcal{R}_{ij} + j \mathcal{X}_{ij} \right) u_j \, dS \tag{31}$$

Analytical formulations for the resistance and reactance are known only for simple geometries like circular or rectangular pistons in an infinite baffle radiating into a semi-sphere. A number of different formulations are given in [6].

8. RADIATED SOUND POWER

Only odd modes contribute to the radiated sound power [6] since even modes cancel themselves out. This is shown in figure 4, where only the odd plate modes can be seen in a microphone signal taken in about 6" distance central to the plate.

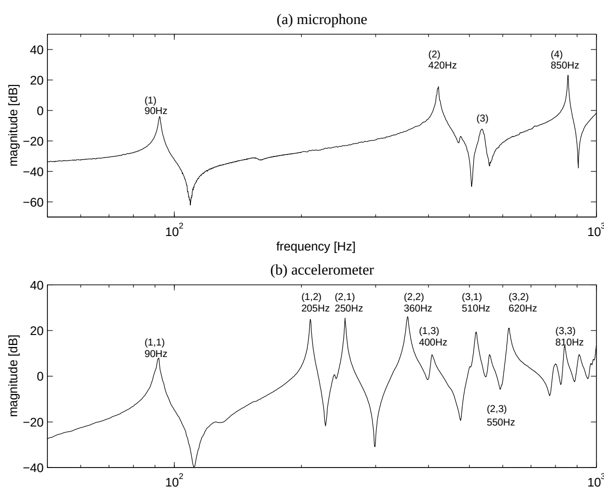

Figure 4: Microphone signal and accelerometer signal. Only the odd plate modes can be found in the acoustic signal.

8.1 Far-field radiation

Using the Green's function (20), the Helmholtz integral equation (28) reduces to

$$p = -2\rho \int_S \ddot{\eta}\, g\, dS \tag{32}$$

Assuming time-harmonic motion of the plate surface $\eta = \hat{\eta} e^{j\omega t}$ and writing the normal acceleration $\ddot{\eta}$ in terms of the surface normal velocity $u = \dot{\eta}$, $\ddot{\eta} = -j\omega u e^{j\omega t}$, equation (32) transforms into the pressure integral

$$\begin{aligned} p(\mathbf{r}) &= j\omega \frac{\rho_\infty}{2\pi} \int_{S_0} u(\mathbf{r}_0) \frac{e^{-jk|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\, dS \\ &= \frac{\rho_\infty}{2\pi} \int_{S_0} \ddot{\eta}(\mathbf{r}_0) \frac{e^{-jk|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\, dS \end{aligned} \tag{33}$$

The vector $\mathbf{r}_0$ is the vector from the origin to the incident point on the active plate surface and $\mathbf{r}$ is the vector from the origin to the observation point. For the far-field, where $\| \mathbf{r} \| \gg \| \mathbf{r}_0 \|$, this pressure integral can be further simplified assuming $|\mathbf{r}-\mathbf{r}_0| \approx \mathbf{r}$. Then Equation (33) further simplifies to

$$\begin{aligned} p(\mathbf{r}) &= j\omega \frac{\rho_\infty}{2\pi} \int\limits_{S_0} u \frac{e^{jk|\mathbf{r}-\mathbf{r}_0|}}{\mathbf{r}}\, dS \\ &= \frac{\rho_\infty}{2\pi} \int\limits_{S_0} \ddot{\eta} \frac{e^{jk|\mathbf{r}-\mathbf{r}_0|}}{\mathbf{r}}\, dS \end{aligned} \tag{34}$$

This equation is usually referred to as the *Rayleigh integral.* Some sources refer to eq. (33) as the Rayleigh integral, but since the Rayleigh integral is always referred to as a far-field formulation and eq. (33) as opposed to eq. (34) is valid throughout the field, it appears reasonable to refer to eq. (34) as the far-field Rayleigh integral.

8.2 The acoustic intensity

The instantaneous acoustic intensity is defined as

$$\mathbf{I} = Re\,[p\mathbf{u}]\,. \tag{35}$$

The acoustic power can then be calculated by

$$P = \int\limits_S \mathbf{I} \cdot \mathbf{n}\, dS \approx \sum_i (\mathbf{I}_i \mathbf{n}_i) \cdot S_i \tag{36}$$

where the latter equation is the approximation by discretizing a semi-sphere and summing up the power radiated through the surface elements of this sphere.

In the case of a forward traveling one-dimensional plane wave, the acoustic intensity can be written as

$$\mathbf{I} = \frac{|p|^2}{2\rho c} \tag{37}$$

and the overall radiated sound power can be written as an integral of the semi-sphere with surface area S.

$$P = \frac{1}{2\rho c} \int\limits_S p^2\, dS \approx \frac{1}{2\rho c} \sum_i \left(p_i^2 \cdot \mathbf{n}_i\right) \cdot S_i \tag{38}$$

Using this formulation for the radiated sound power requires the solution of equation (33) which has to be evaluated for the pressure of each element and each frequency. The assumption of a plane wave is only valid in the very-far-field, so the integral has to be evaluated in a semi-sphere with a very large radius.

$$\begin{aligned} \mathbf{I} &= \frac{1}{2\rho c} \frac{\rho^2}{4\pi^2} \int\limits_{S_0} \ddot{\eta}^2 \left(\frac{e^{jkR'}}{R'}\right)^2 dS \\ &= \frac{\rho}{8\pi^2 c} \int\limits_{S_0} \ddot{\eta}^2 \left(\frac{e^{jkR'}}{R'}\right)^2 dS \end{aligned} \tag{39}$$

This formulation leads to an expression for the overall radiated power

$$P = \frac{\rho}{8\pi^2 c} \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_{S_0} \ddot{\eta}^2 \left(\frac{e^{jkR'}}{R'}\right)^2 dS\ sin\theta\ d\theta\ d\phi \tag{40}$$

where the inner integration is over the area of the vibrating surface and the outer integration over the surface area of the semi-sphere.

The computational requirements for this formulation are very high, especially due to the required high spatial resolution. A real-time implementation evaluating the integrals is currently not possible. Instead, an approximation has to be used.

8.3 Calculation of the acoustic impedance

The influence of a source velocity u at a point (x_0, y_0) on the pressure p at another point (x_1, y_1) is given by the acoustic impedance

$$\mathcal{Z}(x_1, y_1 | x_0, y_0) = \mathcal{Z}_{1|0} = \frac{p(\omega, x_1, y_1)}{u(\omega, x_0, y_0)} \tag{41}$$

Superposing this factors for all source points on the active plate surface leads to a surface integral for the acoustic pressure at a point (x_1, y_1)

$$p(\omega, x_1, y_1) = \int_{S_0} \mathcal{Z}(\omega, x_1, y_1 | x_0, y_0) u(\omega, x_0, y_0)\, dS_0 \quad (42)$$

where $dS_0 = dx_0\, dy_0$ and S_0 being the active surface area of the plate.

For a piston, where all points on the plate surface move in phase and with the same velocity, Morse and Ingard [6] derived the acoustic impedance $\mathcal{Z}$ in terms of resistance $\mathcal{R}$ and reactance $\mathcal{X}$ as

$$\mathcal{R}(ka) = 1 - 4\frac{1 - J_0(ka)}{(ka)^2} \quad (43)$$

$$\mathcal{X}(ka) = \frac{8}{\pi(ka)}\left[1 - \frac{\pi}{2(ka)} \cdot M_0(ka)\right] \quad (44)$$

where (ka) is the dimensionless spatial frequency, the product of the wavenumber $k = \omega/c$ and a characteristic dimension a of the piston. J_0 is the Bessel function of zero oder and M_0 the Struve function of zero order [6].

Assuming a plate motion in terms of the mode shapes, the formulation for the acoustic resistance and reactance of the vibrating plate yields

$$\mathcal{R}_{mn}(x_0, y_0) = \frac{\omega}{2\pi\rho c} \frac{\int_{S_0} \sin(\frac{\pi m x}{a}) \sin(\frac{\pi n y}{b}) \frac{\sin(kR')}{R'}\, dS_0}{\sin(\frac{\pi m x_0}{a}) \sin(\frac{\pi n y_0}{b})} \quad (45)$$

$$\mathcal{X}_{mn}(x_0, y_0) = -\frac{j\omega}{2\pi\rho c} \frac{\int_{S_0} \sin(\frac{\pi m x}{a}) \sin(\frac{\pi n y}{b}) \frac{\cos(kR')}{R'}}{\sin(\frac{\pi m x_0}{a}) \sin(\frac{\pi n y_0}{b})}\, dS_0 \quad (46)$$

This equation cannot be evaluated analytically. Instead, it is assumed that the plate consists of a large number of small pistons and the integration is replaced with a summation over all N pistons, each with a surface area S_i and a mean amplitude of $\overline{\eta}_i$.

$$\mathcal{R}_j = \frac{\omega}{2\pi c} \frac{\sum_{i=0}^{N} \overline{\eta}_i \frac{\sin(kR')}{R'} S_i}{\overline{\eta}_j} \quad (47)$$

$$\mathcal{X}_j = -\frac{j\omega}{2\pi c} \frac{\sum_{i=0}^{N} \overline{\eta}_i \frac{\cos(kR')}{R'} S_i}{\overline{\eta}_j} \quad (48)$$

Rewriting eq. (47) and using the identity $\sum S_i = S_0$ and $k = \omega/c$ the resistance is

$$\mathcal{R}_j = \frac{\omega^2 S_0}{2\pi c^2} \sum_{i=0}^{N} \frac{\overline{\eta}_i}{\overline{\eta}_j} \frac{sin(kR')}{kR'} \quad (49)$$

and using the limit expressing

$$\lim_{R' \to 0} \left(\frac{sin(kR')}{kR'} \right) = 1$$

one can establish the following matrix relating the influence on point i due to an incidence at point j which is called the matrix of radiation resistances and is defined as $\tilde{\mathbf{R}} = (S/2) Re\,[\mathbf{Z}] = (S/2) \cdot (\rho c) \cdot \mathcal{R}$.

$$\tilde{R}_{ij} = \frac{\omega^2 \rho S_i^2}{4\pi c} \frac{\overline{\eta}_j}{\overline{\eta}_i} \frac{sin(kR'_{ij})}{kR'_{ij}} \quad (50)$$

$$\tilde{\mathbf{R}} = \frac{\omega^2 \rho S_0^2}{4\pi c} \cdot \begin{bmatrix} 1 & \frac{\overline{\eta}_2}{\overline{\eta}_1} \frac{sin(kR'_{12})}{kR'_{12}} & \cdots & \frac{\overline{\eta}_N}{\overline{\eta}_1} \frac{sin(kR'_{1N})}{kR'_{1N}} \\ \frac{\overline{\eta}_1}{\overline{\eta}_2} \frac{sin(kR'_{12})}{kR'_{12}} & 1 & & \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\overline{\eta}_1}{\overline{\eta}_N} \frac{sin(kR'_{1N})}{kR'_{1N}} & & & 1 \end{bmatrix} \quad (51)$$

For a piston, where all points are moving in phase with the same amplitude $(\overline{\eta}_j = \overline{\eta}_i)$, this matrix can be simplified further and gets symmetric and positive definite. This matrix of elemental radiation resistances will be used later for estimating the overall radiated sound power.

9. RADIATION MODES

For simple geometries, like the plate considered in the thesis underlying this work [3], mode shapes can be assumed and the radiated power due to these mode shapes can be estimated. The mode shapes have either to be well known or a high number of orthogonal functions has to be used. This approach is only valid when most of the radiated power is radiated due to a small and limited number of mode shapes.

Elliott and Johnson [4] show that the acoustic power output in this case can be written as

$$P(\omega) = \boldsymbol{\eta}^H \tilde{\mathbf{M}}(\omega) \boldsymbol{\eta} \quad (52)$$

where $\tilde{\mathbf{M}}$ is the matrix of the modal radiation resistances and is real, symmetric, and positive definite. The diagonal terms of $\tilde{\mathbf{M}}$ are the self-radiation resistances and the off-diagonal terms are the mutual-radiation resistances. The vector $\boldsymbol{\eta}$ is the vector of the complex structural mode amplitudes, which are to be measured on the system.

Since the matrix $\tilde{\mathbf{M}}$ is real and symmetric, it has an eigenvector/eigenvalue decomposition

$$\tilde{\mathbf{M}} = \mathbf{P}^T \boldsymbol{\Omega} \mathbf{P}. \quad (53)$$

Writing the acoustic power output as

$$P = \boldsymbol{\eta}^H \tilde{\mathbf{M}} \boldsymbol{\eta} = \boldsymbol{\eta}^H \mathbf{P}^T \boldsymbol{\Omega} \mathbf{P} \boldsymbol{\eta} \tag{54}$$

and defining the vector of the transformed acoustic radiation mode $\mathbf{b} = \mathbf{P}\boldsymbol{\eta}$, the acoustic power output in terms of the radiation mode is

$$P = \mathbf{b}^H \boldsymbol{\Omega} \mathbf{b} = \sum_{n=1}^{N} \Omega_n |b_n|^2 \tag{55}$$

which shows that all radiation modes radiate independently of each other. Radiation modes for a rectangular plate and $ka = 0.1$ were shown by Elliott and Johnson [4] and are also plotted in fig. 5. The radiation modes depend on the wavenumber k and therefore on the frequency ω because they are the structural modes transformed by the eigenvalues of the matrix of radiation resistances $\tilde{\mathbf{M}}(\omega)$ which itself depends upon the frequency ω.

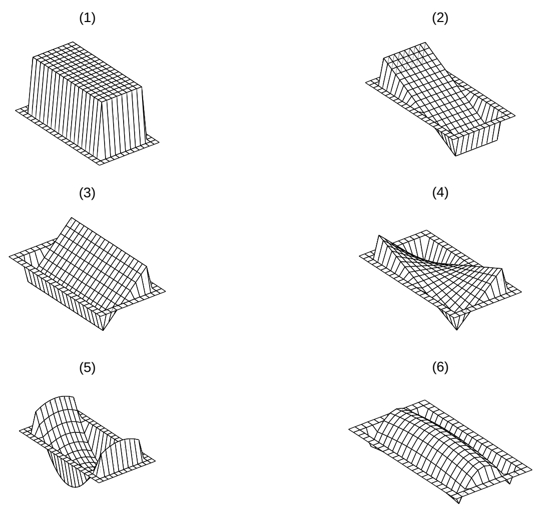

Figure 5: Lower plate radiation modes for $ka = 0.1$ as shown in Elliott and Johnson [4]

9.1 The matrix of radiation resistances

In a formulation in terms of elemental radiators, the acoustic impedance relation for the acoustic pressure can be written as $\mathbf{p} = \mathbf{Z} \cdot \mathbf{u}$, where $\mathbf{Z}$ is a matrix of acoustic impedances. The matrix of radiation resistances $\tilde{\mathbf{R}} = (S/2)Re\,[\mathbf{Z}]$ is then defined as the real part of this matrix of acoustic impedances $\mathbf{Z}$ times half the area of an elemental radiator.

Baumann et.al [1] give a derivation of this matrix by calculating the far-field radiated power which is presented here.

Assuming a separable and time-harmonic motion of the plate surface which can be written in N modes.

$$u = \sum_{i=1}^{N} \hat{u}_i e^{j\omega t} \psi_i = e^{j\omega t} \sum_{i=1}^{N} \hat{u}_i \psi_i . \tag{56}$$

The far-field Rayleigh integral (Eq. (34)) is rewritten for one of these modes. This formulation shows that the relation between the far-field pressure and the surface velocity can be written in terms of a transfer function $H(\omega)$.

$$\begin{aligned} p_i(t) &= \hat{u}_i e^{j\omega t} \underbrace{\frac{j\omega\rho_\infty}{2\pi} \int_S \psi_i e^{-jkr_0}\, dS\, \frac{e^{jkr}}{r}}_{H(\omega)} \\ &= \hat{u}_i e^{j\omega t} \end{aligned} \tag{57}$$

$$\frac{p_i(\omega)}{u_i(\omega)} = H(\omega) \tag{58}$$

In the far-field the pressure wave can be modelled as a forward traveling plane wave, so the instantaneous intensity $\mathbf{I}$ can be calculated according to eq. (37). Integrating the intensity with respect to time gives a formulation for the radiated power. The radiated power per unit area which is derived first, is in the following equations denoted by P_{spec}.

$$\begin{aligned} P_{spec} &= \frac{1}{\rho c} \int_0^\infty p^2\, dt \\ &= \frac{1}{\rho c} \int_0^\infty \left(\sum_{i=1}^{N} p_i(t) \right)^2 dt \end{aligned} \tag{59}$$

(60)

This equation is rewritten applying Parseval's theorem and the vector of modal velocities $\mathbf{w}$ and the vector of modal transfer functions $\mathbf{h}$ are introduced.

$$\begin{aligned} P_{spec} &= \frac{1}{\pi\rho c} \int_0^\infty \left(\sum_{i=1}^{N} p_i(\omega) \right)^2 d\omega \\ &= \frac{1}{\pi\rho c} \int_0^\infty \mathbf{w}^*(\omega) \mathbf{h}^*(\theta,\phi,\omega) \mathbf{h}(\theta,\phi,\omega) \mathbf{w}(\omega)\, d\omega \end{aligned} \tag{61}$$

Integrating the radiated power per unit area over the surface of a sphere of radius R, such that R is in the far-field, gives an integral formulation for the total radiated power.

$$
\begin{aligned}
P &= \int_0^{2\pi}\int_0^{\pi} \frac{1}{\pi\rho c} \int_0^{\infty} \mathbf{w}^* \mathbf{h}^* \mathbf{h} \mathbf{w}\ sin\theta\ d\theta\ d\psi \\
&= \int_0^{\infty} \mathbf{w}^* \tilde{\mathbf{M}} \mathbf{w}\ d\omega \qquad (62)
\end{aligned}
$$

in this formulation, the matrix of radiation impedances $\tilde{\mathbf{M}}$ is defined as

$$
\tilde{\mathbf{M}} = \frac{1}{\pi\rho c} \int_0^{2\pi}\int_0^{\pi} \mathbf{h}^* \mathbf{h}\ sin\theta\ d\theta\ d\psi \qquad (63)
$$

Elliott and Johnson [4] show that the relation between the matrix of radiation resistances and the matrix of modal radiation resistances and therefore between the formulation in mode shapes and the formulation in elemental radiators is given by

$$
\tilde{\mathbf{M}}(\omega) = \mathbf{\Phi}^T \tilde{\mathbf{R}}(\omega) \mathbf{\Phi} \qquad (64)
$$

where the matrix $\mathbf{\Phi}$ relates the structural mode amplitudes $\boldsymbol{\eta}$ to the vector of the elemental velocity amplitudes $\mathbf{w}$

$$
\mathbf{w} = \mathbf{\Phi}\boldsymbol{\eta}. \qquad (65)
$$

The numerical calculation of the matrix of radiation resistances $\tilde{\mathbf{R}}$ can easily be done, even for complicated geometries, the self- and mutual-radiation resistances can then be calculated easily using the transfer matrix $\mathbf{\Phi}$

For the simple geometry of a flat plate the matrix of radiation resistances can be formulated analytically

$$
\tilde{\mathbf{R}} = \frac{\omega^2 \rho S^2}{4\pi c} \begin{bmatrix} 1 & \frac{\sin(kr_{12})}{r_{12}} & \cdots & \frac{\sin(kr_{1I})}{r_{1I}} \\ \frac{\sin(kr_{21})}{r_{21}} & 1 & \cdots & \\ \vdots & & \ddots & \vdots \\ \frac{\sin(kr_{I1})}{r_{I1}} & & \cdots & 1 \end{bmatrix} \qquad (66)
$$

where r_{ij} is the distance from element i to element j. This matrix is clearly symmetric since $r_{ij} = r_{ji}$ and can be compared to eq. (51) which was derived without a far-field assumption and is certainly valid throughout the field. This shows, that the far-field assumption is not necessary, and the matrix of radiation resistances can be used throughout the field.

9.2 Radiation efficiencies

The radiation efficiency is usually defined as [4]

$$
\sigma = P/\rho c S_0 \overline{u}^2 \qquad (67)
$$

where S_0 is the total surface area of the radiator and $\overline{u}^2$ is its space average mean-square velocity.

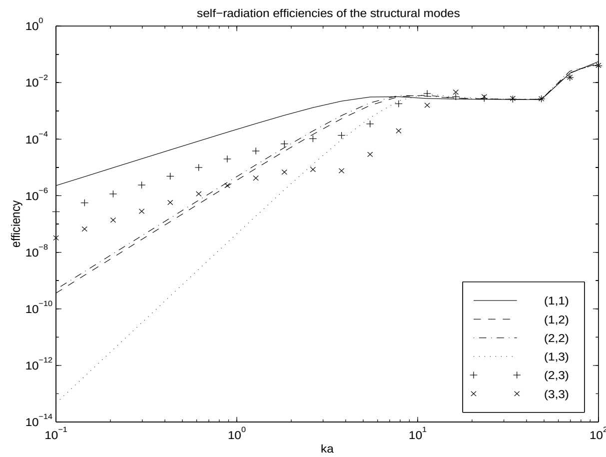

Figure 6: The self-radiation efficiencies of the plate mode shapes

The overall radiated power is given by eq. (52) and the mean square velocity is given by $|\boldsymbol{\eta}|^2/2$. The self-radiation efficiencies can then be written as

$$
\sigma_{ii} = \frac{2}{\rho c S_0} \tilde{M}_{ii} \qquad (68)
$$

and the matrix of radiation efficiencies can then be defined as

$$
\mathbf{\Sigma} = \frac{2}{\rho c S_0} \tilde{\mathbf{M}}. \qquad (69)
$$

Usually it is easier to calculate the matrix $\tilde{\mathbf{R}}$ than the matrix $\tilde{\mathbf{M}}$, so the matrix of radiation efficiencies can be written in terms of the matrix of elemental radiation resistances.

$$
\mathbf{\Sigma} = \frac{2}{\rho c S_0} \mathbf{\Phi}^T \tilde{\mathbf{R}} \mathbf{\Phi} \qquad (70)
$$

The diagonal elements of the matrix of radiation efficiencies $\mathbf{\Sigma}$ are the self-radiation efficiencies of the mode shapes and the off-diagonal elements are the mutual radiation efficiencies.

The radiation efficiencies for the plate in the experimental setup have been calculated using a 10×10 grid of elemental radiators on the plate surface. The result is shown in fig. 6. It can be seen, that for the lower frequencies (lower ka), the $(1,1)$ mode radiates with the best efficiency, and for high frequencies, above about $ka \approx 100$, all structural modes radiated with about the same self-radiation efficiency.

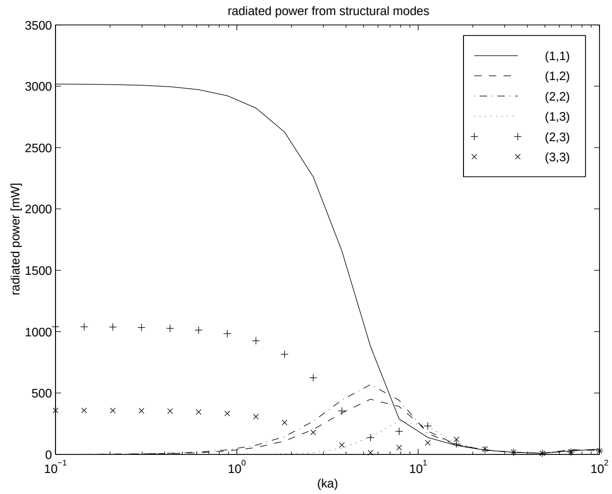

Figure 7: Simulation of the Overall radiated sound power of the single modes from the experimental plate

Using eq. (52) the overall radiated sound power can be calculated. The result for a simulation of the experimental plate, using 10×10 elemental radiators and assuming all modes $(1,1)$ to $(3,3)$ radiate with the same amplitude, is shown in fig. 7. Figure 8 shows the portion of the overall radiated sound power caused by single structural modes. It can be seen, that for $(ka) \lessapprox 10$, the $(1,1)$ mode dominates the radiated power, but for $(ka) \gtrapprox 10$, all structural modes contribute equally to the overall radiated sound power.

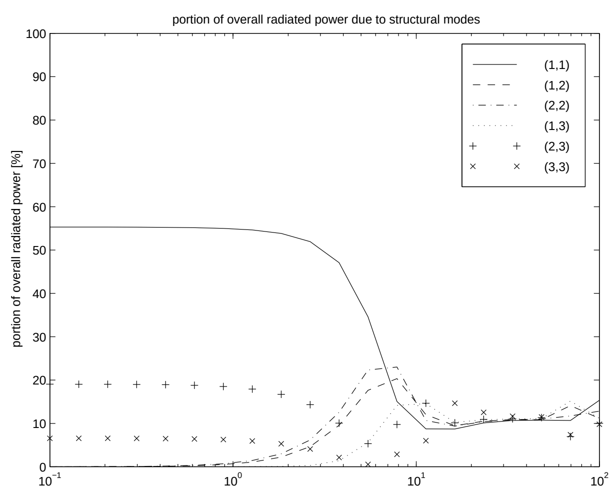

Figure 8: Portion of radiated sound power due to structural modes

Again, it is emphasized that this is only valid for the assumption, that all modes radiate with the same amplitude. It is far more likely, that the amplitude of the $(1,1)$ mode is higher than those of the other modes, and therefore,

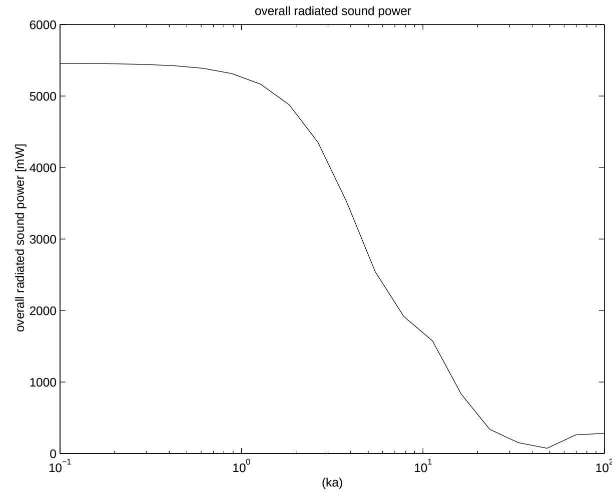

Figure 9: Overall radiated sound power of all modes up to $(3,3)$, depending on (ka). Note the rapid falloff at $(ka) \approx 10$

the $(1,1)$ mode contributes at least the shown part of the overall radiated power and, as can be seen in fig. 9, where the overall radiated sound power falls of rapidly at around $(ka) \approx 10$.

10. FLUID-STRUCTURE COUPLING

In this model, only the reaction force on the structure due to the fluid is modelled, the fluid field itself is not explicitly modelled. Usually, in Active Structural Acoustic Control, there is no need for modelling the acoustic field explicitly since the important parameters, the reaction force and the radiated sound power, can be calculated from parameters local to the plate surface only.

The reaction force of the fluid is implemented as an additional forcing term in the Rayleigh-Ritz model, so the reaction of each mode shape has to be calculated. For a given normalized output function $\mathcal{N} = p/u = \rho c\,(\mathcal{R} + j\mathcal{X})$ the reaction force $F = p\,dS$ can be calculated depending on the mode shapes.

The structural equation of motion is given in the form

$$m_s\ddot{x} + c_s\dot{x} + k_s x = f_{ext} + f_{fluid}. \qquad (71)$$

Assuming a harmonic motion $x = \eta_0 e^{j\omega t}$, the time derivatives are $\dot{x} = j\omega x$ and $\ddot{x} = -\omega^2 x$, the fluid

	w/o FSI		*air*		*water*
mode shape	analytical	Rayleigh Ritz	Experiment	FE 2d/3d	FE 2d/3d
(1,1)	94.6	94.3	90.0	93.8	37.5
(1,2)	215.5	215.1	208.8	213.8	118.1
(2,1)	257.7	257.9	251.9	255.2	149.4
(2,2)	378.6	379.6	353.8	373.3	248.6
(1,3)	416.9	415.7	403.1	413.9	287.8
(3,1)	529.5	527.7	523.1	523.2	384.9
(2,3)	580.0	581.2	553.1	569.8	431.1
(3,2)	650.4	651.4	622.5	638.8	495.0
(3,3)	851.8	850.3	805.0	845.3	705.7

Table 3: The first nine eigenfrequencies of the plate for the plate w/ and w/o fluid loading

reaction force is given in the modal amplitude η_0 by

$$\begin{aligned} f_{fluid} &= \mathcal{Z}(\omega, x_0, y_0)\dot{x}\, dS \\ &= \omega\rho c\eta_0 \left(-\mathcal{X}(\omega) + j\mathcal{R}(\omega)\right)\, dS \end{aligned} \quad (72)$$

Inserting these formulations in the equation of motion yields

$$\begin{aligned} &-\omega^2 m_s\eta_0 + j\omega c_s\eta_0 + k_s\eta_0 \\ &= f_{ext} + j\omega\rho c\eta_0 \left(\mathcal{X}(\omega) - \mathcal{R}(\omega)\right)\, dS \end{aligned} \quad (73)$$

Regrouping so that the terms depending on the modal amplitude η_0 and their derivatives are on the left side and the external forcing terms on the right side of the equation leads to the following formulation

$$\begin{aligned} &-\omega^2 m_s\eta_0 + j\omega(c_s - \rho c\mathcal{R}(\omega)dS)\eta_0 \\ &\quad +(k_s + \omega\rho c\mathcal{X}(\omega)dS)\eta_0 = f_{ext}. \end{aligned} \quad (74)$$

For a fixed frequency ω, the fluid stiffness term is given as $k_{fluid} = \omega\rho c\mathcal{X}(\omega)$.

In equation (74), the structural equation of motion including the reaction of the fluid can be calculated without the need for additional fluid equations. Only quantities local to the plate surface and the characteristic impedance ρc of the fluid are used.

11. CONCLUSIONS

In table 3 eigenfrequencies of the modes up to the $(3,3)$ mode given by different models are listed. The analytical solution is given by Blevins [2] and shown in eq. (3). The Rayleigh Ritz solution was computed using the plate model given by Rodgers [8]. The FE 2d/3d model is an ANSYS model using two-dimensional plate elements for the structure and three-dimensional acoustic elements for the fluid.

It can be seen that the influence of light fluids, like air, can be neglected, but the influence of heavy fluids, especially liquids, can no longer be neglected. The model data for water shows that fluid loading has an higher influence on the lower eigenfrequencies than on higher eigenfrequencies.

Comparable results have been achieved employing an explicit finite-element formulation [3].

The impedance formulation for the impact of fluid loading on the vibration of plates shown in this work, uses only data local to the vibrating surface and can be evaluated using reasonable computational effort. Therefore this formulation seems suitable for the numerical modelling of fluid loading in Active Structural Acoustic Control.

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