Hierarchical Modeling using Dimensional Analysis

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Abstract. Hierarchical modeling allows the aggregation of simple subsystem models into more complex models. It is argued that the combination of hierarchical modeling and dimensional analysis yields a technique that allows the aggregation of submodels without prior assumptions about the connections of the submodels. The method of dimensional analysis is therefore briefly described and then the hierarchical modeling procedure using dimensional analysis is discussed. The example of an gas turbine is used to illustrate the technique of hierarchical modeling using dimensional analysis.

1 Introduction

The modeling of complex physical systems is often a difficult engineering task. Decomposing the complex system into less-complex subsystems, modeling these seperately, and aggregating them into a complex system again is therefore often required in engineering analysis. However, these modeling techniques require a priori knowledge about the coupling of the subsystems in aggregating the subsystems and are therefore not always applicable. Dimensional analysis is a well-known engineering technique, in which a system is modeled in dimensionless groups which are the invariants of the system and which form a minimal set of dimensionless parameters for the system.

This paper presents a method to use a hierarchical modeling technique together with dimensional analysis to allow a formal aggregation of the complex system without prior assumptions of the connections between the subsystems.

In order to achieve this, the technique of dimensional analysis is described first. Then the aggregation of subsystems using dimensional analysis is shown. The example of an aerospace gas turbine illustrates the hierarchical modeling technique using dimensional analysis.

2 Dimensional Analysis

The principle of dimensional homogeneity guarantees that in every possible and correct physical equation the dimensions on the left hand side of the equal sign are identical to those on the right hand side. Due to this property of all possibly correct physical functions $f(x_1, \ldots, x_n) = 0$, the Pi-Theorem of Buckingham [1] holds.

Theorem 1 (Pi-Theorem) *From the existence of a complete and dimensionally homogeneous function f of n physical quantities $x_i \in \mathbb{R}^+$ follows the existence of a dimensionless function F of only $m \leq n$ dimensionless quantities $\pi_j \in \mathbb{R}^+$*

$$f(x_1, \ldots, x_n) = 0 \tag{1}$$

$$F(\pi_1, \ldots, \pi_m) = 0 \tag{2}$$

where $m = n - r$ is the number of dimensional quantities n reduced by the rank r of the dimensional matrix formed by the n dimensional quantities. The dimensionless quantities (also dimensionless products or dimensionless groups) have the form

$$\pi_j = x_{j+r} \prod_{i=1}^{r} x_i^{-\alpha_{(j+r)i}} \qquad j = 1, \ldots, m \tag{3}$$

for $j = 1, \ldots, m \in \mathbb{N}^+$ and with the $\alpha_{ji} \in \mathbb{R}$ as constants.

The restriction to positive values of the dimensional parameters $x_i \in \mathbb{R}^+$ can be satisfied by coordinate transforms and is common in physics. Additionally it can be shown that proofs of the Pi-Theorem impose no restriction on the specific kind of the operator f and are thus valid for all physical equations witout exception.
The dimensionless groups π_j can be derived from the relevance list of dimensional variables. The dimensional matrix (shown in fig. 1 on the left hand side) is formed by n rows for the dimensional variables x_i and up to k columns for the dimensional exponents e_{ij} of the variables x_i in the k base dimensions s_k of the employed unit system. Usually the SI unit system with its seven base dimensions (mass, length, time, temperature, current, amount of substance, and intensity of light) is used so that $k \leq 7$.

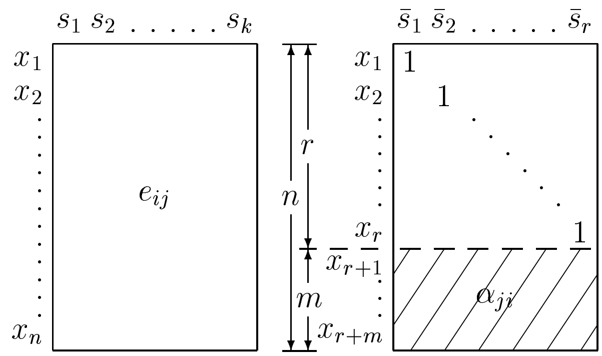

Figure 1: Dimensional matrix [5]

Applying rank preserving operations, the dimensional matrix is converted into an upper diagonal form. The allowed operations include adding multiples of the columns to each other or the exchange of matrix rows. The unknown exponents α_{ji} of the dimensionless products in equation (3) are automatically determined by negation of the values of the resulting matrix elements α_{ji} in the hatched part of the matrix in figure 1.

2.1 Aspects of dimensional modeling

Usually the method of dimensional analysis is used for a single system. Sometimes the systems are so complex that it is desirable to model less complex aspects of the system independently and aggregate the submodels into the complete model of the system afterwards. The less complex models can be found either by splitting the system in physical subsystems or by modeling the different physical phenomena independently, as exemplarily shown in figure 2.

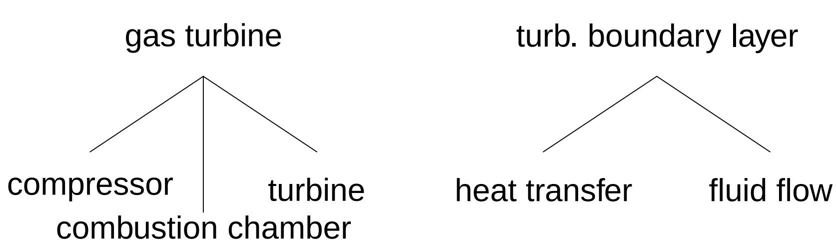

Figure 2: Hierarchical modeling, subsystems (left), subfunctionality (right)

In both cases, dimensional analysis provides a tool to find the coupling parameters necessary to integrate the submodels into the model of the aggregated system, as will be shown in the following.
Figure 3 shows the aggregation of two subsystems (in the simple case) with identical rank and base dimensions of the two subsystems. The two dimensional matrices of the two subsystems are arranged one upon the other and dimensional analysis is performed on the aggregated dimensional matrix. Again the same dimensionless groups π_j that have been found independently for the two subsystems can be found, and additionaly the former base variables of system 2 form additional "coupling numbers" (see section 3.2 for details) in connecting variables from system 2 with the base variables taken from system 1.

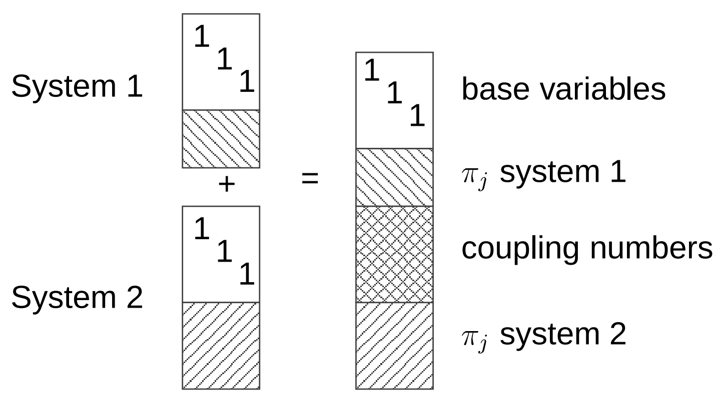

Figure 3: Hierarchical modeling, aggregation of two subsystems with identical rank and base dimensions

In the next section, the number of coupling numbers is determined for this case as well as for the general case for n subsystems where the rank and the base dimensions of the subsystems do not necessarily coincide.

3 Hierarchical Modeling

Hierarchical modeling is usually seen as a functional modeling scheme using subfunctionalities and higher, more abstract levels of modeling with a reduced parameter set as shown in figure 4 for the example of an aerospace gas turbine. In this kind of hierarchical modeling additional parameters for the aggregated system that are not present in the subsystems are usually needed, such as $\dot{Q}_H$ in figure 4. The subsystems are formulated as explicit equations and only the result (i.e. the dependent variable) of the subsystem equations is used in the formulation of the aggregated system.

Coupling conditions

$$\dot{m} = \dot{m}_C = \dot{m}_B = \dot{m}_T$$
$$c_p = c_{p,C} = c_{p,B} = c_{p,T}$$
$$T_2 = T_{2,C} = T_{2,B}$$
$$T_3 = T_{3,B} = T_{3,T}$$

$$\dot{Q}_H = f(\dot{W}_C, \dot{W}_T, \dot{Q}_F)$$

$$\dot{Q}_F = f_B(\dot{m}_B, c_{p,B}, T_{2,B}, T_{3,B})$$

$$\dot{W}_T = f_T(\dot{m}_T, c_{p,T}, T_{3,T}, T_{4,T})$$

$$\dot{W}_C = f_C(\dot{m}_C, c_{p,C}, T_{1,C}, T_{2,C})$$

Figure 4: Functional hierarchical modeling, the functional relations shown are taken from the example of the aerospace gas turbine explained in section 4

In this paper, a different type of hierarchical modeling technique is presented. The higher-level models incorporate all parameters of the submodels (there is no abstraction in the aggregation process). The aggregated system is an accumulation of all the parameters of the subsystems. The number of parameters in the aggregated system can later be reduced after the aggregation step using additional knowledge about the connections between the subsystems as shown in the following sections.

3.1 Two systems with identical rank and base dimensions

Considering a physical system with n_i physical variables and the rank of the dimensional matrix r_i, exactly

$$m_i = n_i - r_i \tag{4}$$

independent dimensionless groups, forming a minimal set of parameters for the system, can be found.

Merging two physical systems, now called *subsystems*, with n_i physical variables and the ranks r_i respectively, into one single system leads to a dimensional matrix with $n = n_1 + n_2$ physical variables and the rank r.

If the ranks and the base dimensions of the two subsystems are identical, the rank of the aggregated system is given by $r = r_1 = r_2$. In this case the number of dimensionless groups for the aggregated system is given by

$$\begin{aligned} m &= n - r \\ &= (n_1 + n_2) - r \\ &= (n_1 - r_1) + (n_2 - r_2) - r + r_1 + r_2 \\ &= (n_1 - r_1) + (n_2 - r_2) + r \\ &= m_1 + m_2 + r \end{aligned} \tag{5}$$

In this example r additional dimensionless groups compared to the sum of the number of dimensional groups from the two subsystems have to be determined. Because of equation (4), there can only be formed $m_1 + m_2$ dimensionless groups in the two subsystems independently, the additional r dimensionless groups have to be formed using physical variables from both subsystems concurrently.

3.2 Two systems with different rank or base dimensions

In the above example, the rank and the base dimensions have been identical. If this is not the case, as shown in figure 5, the identity $r = r_1 = r_2$ does not hold any more.

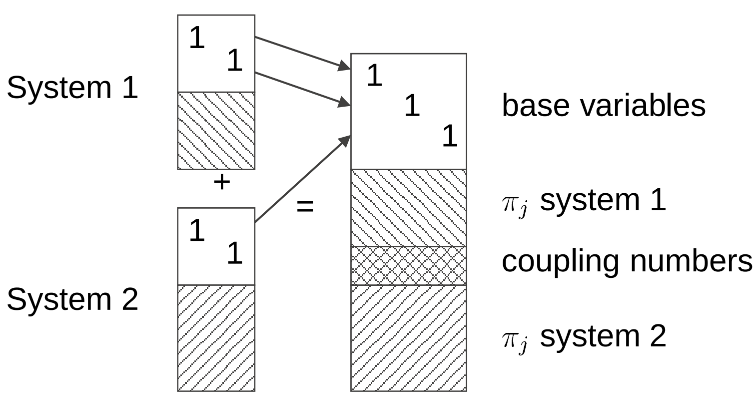

Figure 5: Hierarchical modeling, aggregation of two subsystems with different base dimensions

The number of dimensionless groups for the aggregated system in figure 5 is now

$$\begin{aligned} m &= n - r \\ &= n_1 + n_2 - r \\ &= (n_1 - r_1) + (n_2 - r_2) - r + r_1 + r_2 \\ &= m_1 + m_2 + (r_1 + r_2 - r) \end{aligned} \tag{6}$$

In this case, there are $(r_1 + r_2 - r)$ additional dimensionless numbers that again cannot be formed from physical variables from one single subsystem only. These numbers that consist of physical variables from different subsystems are henceforth called *coupling numbers*.

3.3 Arbitrary number of systems with different rank and/or base dimensions

The general case of t subsystems each consisting of n_i physical variables and posessing rank r_i and m_i dimensionless groups respectively leads to an aggregated system with m dimensionless groups with

$$\begin{aligned} m &= \sum_{i=1}^{t} n_i - r \\ &= \sum_{i=1}^{t} (n_i - r_i) - r + \sum_{i=1}^{t} r_i \\ &= \sum_{i=1}^{t} m_i + \left(\sum_{i=1}^{t} r_i - r \right) \end{aligned} \tag{7}$$

The minimum number m_c of coupling numbers that consist of physical variables of different subsystems is given by

$$\begin{aligned} m_c &= \sum_{i=1}^{t} r_i \ - r \\ &= r \cdot \sum_{i=1}^{t} \frac{r_i}{r} \ - 1 \end{aligned} \tag{8}$$

and depends therefore only on the relative ranks ${}^{r_i}/_{r}$ of the subsystems and the rank of the aggregated system.

Since there exists an infinite number of sets of dimensionless numbers for a given physical problem, there also exists an infinite number of sets of dimensionless groups with more than m_c coupling numbers (i.e. dimensionless groups formed with variables from different subsystems), but there exists no set with less than m_c coupling numbers.

For a given physical system, the number of coupling numbers depends on the choice of the subsystems. But for a given set of subsystems, a minimal number of coupling numbers m_c exists and is based on the ranks of the subsystems only.

4 Example: gas turbine

The principle of hierarchical modeling using dimensional analysis is illustrated using the example of an aerospace gas turbine (fig. 6). The thermodynamical modeling of gas turbines can be found in any reference book on gas turbines or on thermodynamics, such as in Frohn [2]. To illustrate the method of hierarchical modeling, a top-down approach is used. First the gas turbine is viewed as a black box (figure 7) and then the model is gradually refined.

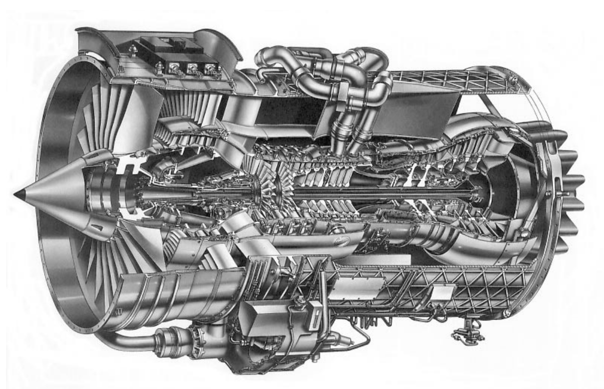

Figure 6: BR715 aerospace gas turbine [3]

The black-box model in figure 7 simplifies the aerospace gas turbine into a process that transforms thermal power $\dot{Q}_F$, supplied in the form of fuel, into the shaft power $\dot{W}_S$ and waste heat $\dot{Q}_H$. The functional modeling of this black-box model of the aerospace gas turbine has the form $f(\dot{Q}_F, \dot{W}_S, \dot{Q}_H) = 0$.

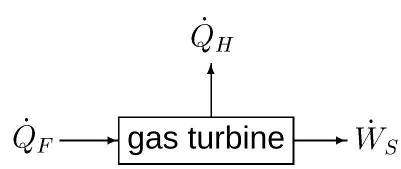

Figure 7: Black box model of an aerospace gas turbine

The variables $\dot{W}_S, \dot{Q}_F,$ and $\dot{Q}_H$ form the relevance list of this black-box model. With this relevance list the dimensional matrix given in table 1 can be established.

	[M]	[L]	[T]		[W]
$\dot{Q}_F$	1	2	-3	$\Rightarrow$	1
$\dot{Q}_H$	1	2	-3		1
$\dot{W}_S$	1	2	-3		1

Table 1: Dimensional matrix for the aerospace gas turbine black-box model

In this example, the rank of the dimensional matrix is equal to 1 and therefore only one independent base dimension for the problem exists. Here the base dimension is work W, written in the SI unit $[W] = [kg]\ [m]^2 / [s]^3$. Transforming the dimensional matrix as shown in table 1 on the right hand side, the two dimensionless groups for the example can be directly taken from the dimensional matrix.

$$\pi_1 = \dot{W}_S/\dot{Q}_F \quad (9)$$

$$\pi_2 = \dot{Q}_H/\dot{Q}_F \quad (10)$$

The similarity function for the black-box model $F(\pi_1, \pi_2) = 0$ can be found employing a simple thermodynamical analysis. The dimensionless groups (eq. (9) and (10)) are for the aerospace gas turbine usually denoted as the efficiency $\eta = \pi_1$ and the loss factor $\Phi = \pi_2$ respectively. The relation between these parameters, i.e. the similarity function, is given by $\eta = 1 - \Phi$, as resulting from the thermodynamical analysis . The implicit form $F(\pi_1, \pi_2) = 0$ of this similarity function is then given by the equation $F(\eta, \Phi) = \eta + \Phi - 1 = 0$.

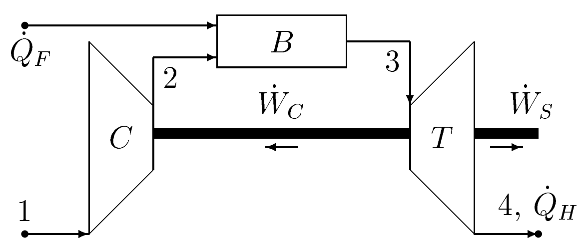

Figure 8: Component scheme of gas turbine [2]

In the next step, the aerospace gas turbine is modeled in the three subsystems compressor *C*, combustion chamber ("burner") *B* and the turbine *T*, as shown in figure 8.

4.1 Compressor

The air soaked into the gas turbine at the intake is compressed to allow the burning of the fuel in the combustion chamber. The compression of the air is done in the compressor which is driven by energy taken from the turbine section and transmitted by the turbine shaft.

The relevant parameters for the compressor are given in table 2 and the according dimensional matrix is shown in table 3.

$\dot{m}_C$	mass flow
$c_{p,C}$	specific heat ratio
$T_{1,C}$	compressor inlet temperature
$T_{2,C}$	compressor outlet temperature
$\dot{W}_C$	compressor power

Table 2: Relevance list of the compressor

	[M]	[L]	[T]	[θ]	SI-Units		$[m_1]$	$[m_2]$	$[m_3]$
$\dot{m}_C$	1		-1		kg/s		1	0	0
$c_{p,C}$		2	-2	-1	$J/kg\,K$	$\Rightarrow$	0	1	0
$T_{1,C}$				1	K		0	0	1
$\dot{W}_C$	1	2	-3		W		1	1	1
$T_{2,C}$				1	K		0	0	1

Table 3: Dimensional matrix of the compressor

The number of parameters in the relevance list (table 2) is euqal to 5 and the rank of the dimensional matrix (table 3) is equal to 3. Therefore $5-3=2$ dimensionless groups for the compressor can be determined.

$$\pi_{C,1} = \frac{\dot{W}_C}{\dot{m}_C\, c_{p,C}\, T_{1,C}} \tag{11}$$

$$\pi_{C,2} = \frac{T_{2,C}}{T_{1,C}} \tag{12}$$

4.2 Combustion chamber

In the combustion chamber fuel is injected into the fluid stream and ignited. The chemical energy of the fuel is converted into heat (higher temperature of the fluid). According to the compressor subsystem, the relevance list (table 4) and the dimensional matrix (table 5) for the combustion chamber can be found. Modeling the subsystems, no assumptions about the connections of the subsystem are made. Therefore the mass flow, temperatures, specific heat ratio, etc. are presumed to be different for each subsystem.

$\dot{m}_B$	mass flow
$c_{p,B}$	specific heat ratio
$T_{2,B}$	chamber inlet temperature
$T_{3,B}$	chamber outlet temperature
$\dot{Q}_F$	thermal power

Table 4: Relevance list of the combustion chamber

	[M]	[L]	[T]	[θ]	SI-Units		[m_1]	[m_2]	[m_3]
$\dot{m}_B$	1		-1		kg/s		1	0	0
$c_{p,B}$		2	-2	-1	$J/kg\,K$	$\Rightarrow$	0	1	0
$T_{2,B}$				1	K		0	0	1
$\dot{Q}_F$	1	2	-3		W		1	1	1
$T_{3,B}$				1	K		0	0	1

Table 5: Dimensional matrix of the combustion chamber

The dimensionless parameters for the combustion chamber subsystem are then determined using the transformed dimensional matrix in table 5 on the right hand side. The two dimensionless groups are

$$\pi_{B,1} = \frac{\dot{Q}_F}{\dot{m}_B\ c_{p,B}\ T_{2,B}} \qquad (13)$$

$$\pi_{B,2} = \frac{T_{3,B}}{T_{2,B}} \qquad (14)$$

4.3 Turbine

The heated gas is expanded in the turbine stage of the aerospace gas turbine. In this section, the heat from the hot fluid is converted into mechanical shaft energy which is partly used to drive the compressor.

The relevance list and the dimensional matrix for the turbine subsystem are given in table 6 and table 7 respectively.

$\dot{m}_T$	mass flow
$c_{p,T}$	specific heat ratio
$T_{3,T}$	turbine inlet temperature
$T_{4,T}$	turbine outlet temperature
$\dot{W}_T$	turbine power

Table 6: Relevance list of the turbine subsystem

	[M]	[L]	[T]	[θ]	SI-Units		[m_1]	[m_2]	[m_3]
$\dot{m}_T$	1		-1		kg/s		1	0	0
$c_{p,T}$		2	-2	-1	$J/kg\,K$	$\Rightarrow$	0	1	0
$T_{3,T}$				1	K		0	0	1
$\dot{W}_T$	1	2	-3		W		1	1	1
$T_{4,T}$				1	K		0	0	1

Table 7: Dimensional matrix of the turbine subsystem

According to the relevance list (table 6) and the dimensional matrix (table 7) for the turbine stage, dimensional analysis yields two dimensionless groups.

$$\pi_{T,1} = \frac{\dot{W}_T}{\dot{m}_T\ c_{p,T}\ T_{3,T}} \tag{15}$$

$$\pi_{T,2} = \frac{T_{4,T}}{T_{3,T}} \tag{16}$$

4.4 Aggregation of the subsystems

The three subsystems compressor, combustion chamber, and turbine can be integrated into the overall system of the aerospace gas turbine. The three subsystems have $m_C = 2$, $m_B = 2$, and $m_T = 2$ dimensionless parameters respectively. The aggregated system has $n = n_C + n_B + n_T = 5 + 5 + 5 = 15$ dimensional variables and the dimensional matrix has the rank $r = 3$. Therefore, the aerospace gas turbine has $m = n - r = 15 - 3 = 12$ dimensionless groups. Only $\sum m_i = 6$ dimensionless groups can be built using only variables from the same subsystem, the other $m - \sum m_i = 12 - 6 = 6$ dimensionless groups must be built using dimensional variables from different subsystems. Therefore the minimal number of coupling numbers is 6.

Combining the relevance lists of the three subsystems (table 2, 4, and 6) according to the methodology described in figure 3 yields the relevance list of the aerospace gas turbine as shown in table 8.

$\dot{m}_C$	compressor mass flow
$c_{p,C}$	compressor specific heat ratio
$T_{1,C}$	compressor inlet temperature
$T_{2,C}$	compressor outlet temperature
$\dot{W}_C$	compressor power
$\dot{m}_B$	chamber mass flow
$c_{p,B}$	chamber specific heat ratio
$T_{2,B}$	chamber inlet temperature
$T_{3,B}$	chamber outlet temperature
$\dot{Q}_F$	thermal power
$\dot{m}_T$	turbine mass flow
$c_{p,T}$	turbine specific heat ratio
$T_{3,T}$	turbine inlet temperature
$T_{4,T}$	turbine outlet temperature
$\dot{W}_T$	turbine power

Table 8: Relevance list of the gas turbine

	[M]	[L]	[T]	[θ]	SI-Units		$[m_1]$	$[m_2]$	$[m_3]$
$\dot{m}_C$	1		-1		kg/s		1	0	0
$c_{p,C}$		2	-2	-1	$J/kg\,K$		0	1	0
$T_{1,C}$				1	K		0	0	1
$\dot{W}_C$	1	2	-3		W		1	1	1
$T_{2,C}$				1	K		0	0	1
$\dot{m}_B$	1		-1		kg/s		1	0	0
$c_{p,B}$		2	-2	-1	$J/kg\,K$		0	1	0
$T_{2,B}$				1	K	$\Rightarrow$	0	0	1
$\dot{Q}_F$	1	2	-3		W		1	1	1
$T_{3,B}$				1	K		0	0	1
$\dot{m}_T$	1		-1		kg/s		1	0	0
$c_{p,T}$		2	-2	-1	$J/kg\,K$		0	1	0
$T_{3,T}$				1	K		0	0	1
$\dot{W}_T$	1	2	-3		W		1	1	1
$T_{4,T}$				1	K		0	0	1

Table 9: Dimensional matrix of the gas turbine

The dimensional matrix (eq. 9) for the relevance list in table 8 yields the 12 dimensionless groups for the aerospace gas turbine according to figure 1.

$$
\begin{aligned}
\pi_1 &= \frac{\dot{W}_C}{\dot{m}_C\; c_{p,C}\; T_{1,C}} = \pi_{C,1} & \pi_2 &= \frac{T_{2,C}}{T_{1,C}} = \pi_{C,2} \\
\pi_3 &= \frac{\dot{m}_B}{\dot{m}_C} & \pi_4 &= \frac{c_{p,B}}{c_{p,C}} \\
\pi_5 &= \frac{T_{2,B}}{T_{1,C}} & \pi_6 &= \frac{\dot{Q}_F}{\dot{m}_B\; c_{p,B}\; T_{2,B}} = \pi_{B,1} \\
\pi_7 &= \frac{T_{3,B}}{T_{2,B}} = \pi_{B,2} & \pi_8 &= \frac{\dot{m}_T}{\dot{m}_C} \\
\pi_9 &= \frac{c_{p,T}}{c_{p,C}} & \pi_{10} &= \frac{T_{3,T}}{T_{1,C}} \\
\pi_{11} &= \frac{\dot{W}_T}{\dot{m}_T\; c_{p,T}\; T_{3,T}} = \pi_{T,1} & \pi_{12} &= \frac{T_{4,T}}{T_{3,T}} = \pi_{T,2}
\end{aligned}
\qquad (17)\text{-}(28)
$$

The coupling numbers π_5 and π_{10} are thermal similaritiy numbers considering temperatures at states that are separated by processes. Using the transformation [4]

$$\hat{\pi}_j = \prod_{k=1}^{m} \pi_k^{\beta_{jk}} \qquad j = 1, \ldots, m \tag{29}$$

with the matrix of the transformation exponents β_{jk}

$$\beta_{jk} = \left[\begin{array}{ccccccccccccccc}
+1 & & & & & & & & & & & & & & \\
 & +1 & & & & & & & & & & & & & \\
 & & +1 & & & & & & & & & & & & \\
 & & & +1 & & & & & & & & & & & \\
 & -1 & & & +1 & & & & & & & & & & \\
 & & & & & +1 & & & & & & & & & \\
 & & & & & & +1 & & & & & & & & \\
 & & & & & & & +1 & & & & & & & \\
 & & & & & & & & +1 & & & & & & \\
 & -1 & & & & & -1 & & & +1 & & & & & \\
 & & & & & & & & & & +1 & & & & \\
 & & & & & & & & & & & +1 & & & \\
 & & & & & & & & & & & & +1 & & \\
 & & & & & & & & & & & & & +1 & \\
 & & & & & & & & & & & & & & +1
\end{array}\right] \quad (30)$$

an equivalent set of dimensionless groups can be found where the dimensionless numbers $\hat{\pi}_1, \ldots, \hat{\pi}_4, \hat{\pi}_6, \ldots, \hat{\pi}_9, \hat{\pi}_{11}, \ldots, \hat{\pi}_{15}$ are identical to the respective dimensionless numbers in the original set (eq. (17)-(28)). The two altered dimensionless groups are

$$\hat{\pi}_5 = \frac{T_{2,B}}{T_{2,C}} \quad (31)$$

$$\hat{\pi}_{10} = \frac{T_{3,T}}{T_{3,B}} \quad (32)$$

The number of dimensionless groups can be substantially reduced by modeling the process of the aerospace gas turbine, i.e. by making assumptions about the connections between the subsystems.

The thermodynamic process underlying figure 8 is ideally assumed as adiabatic, reversible compression ($1 \rightarrow 2$), isobaric heating ($2 \rightarrow 3$), adiabatic-reversible expansion ($3 \rightarrow 4$), and isobaric cooling ($4 \rightarrow 1$). The connections between the subsystems are modeled as lossless, i.e. mass flow and the specific heat ratio are equal for all subsystems, and the temperatures of the connected outlets and inlets are equal ($T_{2,C} = T_{2,B}, T_{3,B} = T_{3,T}$). From these considerations follows

$$\begin{aligned} \hat{\pi}_3 &= 1 & \hat{\pi}_4 &= 1 \\ \hat{\pi}_5 &= 1 & \hat{\pi}_8 &= 1 \\ \hat{\pi}_9 &= 1 & \hat{\pi}_{10} &= 1 \end{aligned} \quad (33)\text{-}(38)$$

and the set of dimensionless groups can be reduced by these 6 groups which are constant and can therefore not contribute to the modeling of the gas turbine system under the above stated assumptions.

The same result can be found by applying the restriction of lossless connections in the relevance list of the gas turbine and omitting the surplus physical variables, but the procedure shown above is more general, since the values of the coupling numbers could be measured experimentally.

Should the connections between the subsystems not be ideal, then these dimensionless groups (eq. (33)-(38)) are no longer equal to one, or not even constant any more, but account now for the losses on the connections between the subsystems.
Hierarchical modeling using dimensional analysis allows a formal aggregation of the subsystems without such restrictions on the connections between the subsystems. Engineering knowledge about the connections can then be integrated in the assessment of the dimensionless groups of the aggregated system. The coupling numbers found by hierachical modeling account for non-ideal connections between the different subsystems.

5 Summary and Outlook

In this paper, the formal method of hierarchical modeling, the modeling of independent subsystems and the combination into an aggregated system has been shown. Using the example of an aerospace gas turbine, the method has been investigated and it has been shown that the dimensionless groups of the subsystems are preserved. Additional dimensionless numbers, so-called coupling numbers, which account for the connections between the subsystems, have been found straightforward by aggregating the dimensional matrices of the subsystems according to the proposed methodology.
Further work will be done in extending this work onto a functional hierarchy. The hierarchical modeling of subfunctionalities has not been discussed in this paper but shows the same promising features as the hierarchical modeling using subsystems. It seems that not only the dimensionless groups can be preserved, but also that the functional relationships for an aggregated system will have to integrate the functional relationships of all the subsystems.

References

[1] E. Buckingham. On physically similar systems: Illustration of the use of dimensional equations. *Phys. Review*, 4:345–376, 1914.

[2] Arnold Frohn. *Einführung in die Technische Thermodynamik*. Akademische Verlagsgesellschaft, Wiesbaden, 1977.

[3] Rolls Royce Deutschland GmbH. http://www.rolls-royce.de/, Oct. 2000.

[4] Juri Pawlowski. *Die Ähnlichkeitstheorie in der physikalisch-technischen Forschung, Grundlagen und Anwendung*. Springer, Berlin, 1971.

[5] Stephan Rudolph. On Topology, Size and Generalization of Non-linear Feed-Forward Neural Networks. *Neurocomputing*, 16(1):1–22, July 1997.